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The Matrix Algebras of Continuous Groups with Antilinear Operations

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Abstract The corepresentation theory of continuous groups is presented without the assumption that the subgroup G of the group with antilinear operations is unitary. Continuous groups of the form: $G + a_0 G$ are defined, where G denotes a linear Lie group and a_0 denotes an antilinear operation which fulfils the condition $a_0^2 = \pm 1$. The matrix algebras connected with the groups $G + a_0 G$ are defined. The structural constants of these algebras fulfill the conditions following from the Jacobi identities. Applications are presented to the groups $G = SU(d)$, $d = 1, 2, \ldots,$ for $a_0 = K$, the complex conjugation operation, and to the group $SL(2, C)$ for $a_0 = K$ or Θ, the time-reversal operation.

Keywords Continuous groups · Antilinear operations · Corepresentations · Lie groups and algebras

1 Introduction

The theory of corepresentations of non-unitary groups $\mathcal{G} = G + a_0 G$, where G denotes a unitary group and a_0 is an antiunitary element, was formulated by Wigner [32], to whom belong the first applications of corepresentations in quantum mechanics. Space groups with antiunitary operations and their corepresentations subsequently found important applications in solid state physics.

Wigner's theory of corepresentations was elaborated by a number of authors to the form of a powerful tool for investigations of physical properties of crystals and of magnetic crystals [1, 2, 4, 6, 28, 29]. It was applied in the investigations of symmetry changes at commensurate and incommensurate continuous magnetic phase transitions [11–13, 20], and to the problem of magnetocrystalline anisotropy of ferromagnetic crystals [14].

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It was shown by Birman [2] that a non-unitary symmetry group can intervene in the classical description of a crystal in a state of thermodynamic equilibrium. The non-unitary group of the type $G + KG$, where G is a space group, and where K denotes the operation of complex conjugation, constitutes the complete symmetry group of the crystal-lattice dynamic problem. This group plays the basic role in establishing the one-to-one correspondence between vibration frequencies and irreducible corepresentations (coirreps). The application of the non-unitary group $G + KG$ to a description of lattice vibrations elaborated in [2], opened the way for a further development in this field, made by Kovalev [23–28] and by Kovalev and Gorbanyuk [29]. These authors formulated another method of demonstrating that there exists the one-to-one correspondence between coirreps and frequencies of crystal lattice vibrations. As a subsequent step in the exploration of the importance of the non-unitary groups, Kovalev and Gorbanyuk [29], generalized Wigner-Eckart theorem [31] to systems described by magnetic space groups (see [15, 16]).

The corepresentation theory was originally formulated for the case when the subgroup G of the group $G + a_0G$ is unitary [32]. The group $G + a_0G$ then is called a non-unitary group [4], and the element a_0 is an antiunitary operation. The name antiunitary, which was assigned to the antilinear operations of complex conjugation and of time reversal draws from the fact that when the bilinear product of basis functions is Hermitian, any antiunitary operation is equal to the product of the operation of complex conjugation with some unitary operation [32]. The name antiunitary does not seem to be appropriate when those operations are applied to linear operations which are not unitary, for example to the operations of the group $SL(2,C)$. The modification of group representation theory leading to corepresentations is conditioned by the antilinear character of the operations of complex conjugation or time reversal.

In Sect. 2 we will present the theory of corepresentations without making the assumption that the subgroup G of the group $G + a_0G$ is unitary. The formulas of the corepresentation theory with unitary groups G can be obtained from this presentation. Groups of the type $\mathcal{G} = G + a_0G$ will be considered, consisting of the subgroup G which is a group of linear operations and of the coset a_0G, consisting of products of an antilinear operation a_0 with the linear operations belonging to G. The element a_0 itself, in general can be a product of an antilinear operation A with a linear operation g_L^0, which does not belong to the subgroup G. However, the element g_L^0 has to be of such a type that we have $(Ag_L^0)^2 \in G$. In particular we can have g_L^0 equal to the unit element $\mathbf{1}$.

An attempt at an extension of Wigner's investigations of continuous groups with antilinear operations will be presented in Sect. 3. For a certain type of continuous groups with antilinear operations, matrix algebras with the commutator product will be defined. This will be done for continuous groups $G + a_0G$, in which G is a linear Lie group, which need not be unitary, and the antilinear element a_0 fulfills the condition $a_0^2 = \pm 1$. The matrix algebras connected with the thus specified groups $G + a_0G$ can be defined in a way which is analogous to that for linear Lie groups. The parametrization of the groups $G + a_0G$ where G is a linear Lie group requires carefull attention. We will accept the following solution of this problem: When a linear Lie group G depends on n essential parameters the coset a_0G depends on $n + 1$ essential parameters—the n parameters of the Lie group and an additional parameter which in general is required for completing the matrix algebra. The required basis element of the matrix algebra is connected with the antilinear element a_0. The matrix algebras of the groups $SU(d) + KSU(d)$, $d = 1, 2, \dots$, and of the groups $SL(2,C) + a_0SL(2,C)$, where a_0 is the complex conjugation operation or the time-reversal operation, will be calculated in Sects. 4, 5, 6 and 7. A part of the results contained in this paper was presented in [21, 22].

2 The Corepresentation Theory of Continuous Groups

In this section we are indebted to the presentations of the corepresentation theory for magnetic space groups by Bradley and Cracknell [4], and Kovalev and Gorbanyuk [29].

In the applications of corepresentation theory in quantum mechanics [32], the antiunitary element a_0 was the time-reversal operation, multiplied by a proper or improper rotation element, represented by a unitary matrix. When the subgroup G need not be unitary, it seems to be misleading to call the group $G + a_0G$ a non-unitary group, and we will not use this name.

Let G be a continuous group of linear transformations which need not be unitary. We define the group

$$\mathcal{G} = G + a_0 G \tag{1}$$

in which in general the operation a_0 is a product of an antilinear operation with a linear operation, which does not belong to the subgroup G. As the product of any two elements of the coset a_0G has to belong to G, we must have $a_0^2 \in G$.

Let Γ be an irreducible representation (irrep) of the group G, of dimension d, and let φ_i, $i = 1, \dots, d$, be its basis functions. For any element $g \in G$, we then have

$$g\varphi_i = \sum_{j=1}^{d} \Delta(g)_{ji}\varphi_j \quad \text{or} \quad g\varphi = \tilde{\Delta}(g)\varphi \tag{2}$$

where $\Delta(g)$ is the representation matrix, φ is the column matrix constructed from the basis functions $\varphi_1, \dots, \varphi_d$, and $\tilde{\Delta}(g)$ is the transposed matrix. The action of an antilinear operation a_0 on a linear combination of functions φ_i is defined by

$$a_0 \sum_{i=1}^{d} c_i \varphi_i = \sum_{i=1}^{d} c_i^* a_0 \varphi_i \tag{3}$$

where c_i are complex numbers, and $*$ denotes complex conjugation.

The action of the antilinear element a_0 on the basis functions φ_i leads to another set of functions ϕ_i,

$$a_0\varphi_i = \phi_i; \quad i = 1, \dots, d \tag{4}$$

We consider this transformation as an endomorphism of the space which is spanned by the functions φ_i (see in this respect Chap. V in [3]). The column matrix constructed from the functions ϕ_i, $i = 1, 2, \dots, d$ will be denoted by ϕ. The action of $g \in G$ on ϕ is given by

$$g\phi = ga_0\varphi = a_0(a_0^{-1}ga_0)\varphi = a_0\tilde{\Delta}(a_0^{-1}ga_0)\varphi = \tilde{\Delta}^*(a_0^{-1}ga_0)\phi \tag{5}$$

where the last equality is connected with the antilinear character of a_0. From (2) and (5) we obtain

$$g\begin{pmatrix}\varphi\\ \phi\end{pmatrix} = \left(\begin{array}{c|c}\tilde{\Delta}(g) & 0\\ \hline 0 & \tilde{\Delta}^*(a_0^{-1}ga_0)\end{array}\right)\begin{pmatrix}\varphi\\ \phi\end{pmatrix}; \quad \forall g \in G \tag{6}$$

We now define the matrix $\overline{\Delta}(g)$ in the representation $\overline{\Gamma}$, by

$$\overline{\Delta}(g) = \Delta^*(a_0^{-1}ga_0); \quad \overline{\Delta}(g) \in \overline{\Gamma} \tag{7}$$

where $\overline{\Delta}(g)$ is a matrix representative of $g \in G$, in the representation $\overline{\Gamma}$ of G. This equation defines the representation $\overline{\Gamma}$ of G.

Let a be any element of a_0G, say, a_0g. We then obtain

$$a\varphi = a_0 g\varphi = a_0\tilde{\Delta}(g)\varphi = \tilde{\Delta}^*(g)\phi = \tilde{\Delta}^*(a_0^{-1}a)\phi \tag{8}$$

where (2) and (4) and the antilinear character of a_0 have been used. We next obtain

$$a\phi = aa_0\varphi = \tilde{\Delta}(aa_0)\varphi \tag{9}$$

owing to $aa_0 \in G$. From (8) and (9) we obtain the expression

$$a\begin{pmatrix}\varphi\\ \phi\end{pmatrix} = \left(\begin{array}{c|c} 0 & \tilde{\Delta}^*(a_0^{-1}a) \\ \hline \tilde{\Delta}(aa_0) & 0 \end{array}\right)\begin{pmatrix}\varphi\\ \phi\end{pmatrix} \tag{10}$$

If $a = ga_0$, and $g = aa_0^{-1}$, the same formula is obtained, since we have

$$a\varphi = (ga_0)\varphi = \tilde{\Delta}^*(a_0^{-1}ga_0)\phi = \tilde{\Delta}^*(a_0^{-1}a)\phi$$

and

$$a\phi = aa_0\varphi = \tilde{\Delta}(aa_0)\varphi$$

which are (8) and (9), respectively. Equations (6) and (10) demonstrate the invariance of the space spanned by the functions φ_i and ϕ_i, $i = 1, \ldots, d$, under the group $\mathcal{G}$. From (6) and (10) we obtain the matrices

$$D(g) = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta^*(a_0^{-1}ga_0) \end{array}\right); \quad \forall g \in G \tag{11}$$

and

$$D(a) = \left(\begin{array}{c|c} 0 & \Delta(aa_0) \\ \hline \Delta^*(a_0^{-1}a) & 0 \end{array}\right); \quad a = a_0 g \text{ or } a = ga_0,\ \forall g \in G \tag{12}$$

The sets of matrices in (11) and (12) form the *corepresentation* $D\Gamma$ *of the group* $\mathcal{G}$, derived from the representation Γ, with the matrices $\Delta(g)$ of the subgroup G. This corepresentation may be reducible. The corepresentation matrices obey the following set of equations [32]:

$$\begin{aligned} &D(g_1)D(g_2) = D(g_1g_2); \quad \forall g_1, g_2 \in G \\ &D(g)D(a) = D(ga); \quad \forall g \in G, \text{ and } \forall a \in a_0G \\ &D(a)D^*(g) = D(ag); \quad \forall g \in G, \text{ and } \forall a \in a_0G \\ &D(a_1)D^*(a_2) = D(a_1a_2); \quad \forall a_1, a_2 \in a_0G \end{aligned} \tag{13}$$

These are established by examining the action of the respective products of elements g and a on the basis functions, when the antilinear character of the elements a is taken into account. Because of the last two equalities, the mapping $\mathcal{G} \to D\Gamma$ is not a homomorphism.

It can be shown that there is no ambiguity in the assignment of the corepresentation $D\Gamma$, derived from the representation Γ, to the group $\mathcal{G}$. Different choices of a_0 in the definition of $\mathcal{G}$ lead to equivalent corepresentations [4, 32].

2.1 The Equivalence of Two Corepresentations

Performing the basis transformation with a nonsingular transformation S,

$$\tilde{S}\chi = \chi', \quad \text{with} \quad \tilde{\chi} = (\varphi, \phi) \tag{14}$$

where φ and ϕ are given in (6), we obtain

$$g\chi' = \tilde{D}'(g)\chi', \quad \text{hence} \quad D'(g) = S^{-1}D(g)S \tag{15}$$

and

$$a\chi' = \tilde{D}'(a)\tilde{S}\chi, \quad \text{or} \quad a\chi' = a\tilde{S}\chi = \tilde{S}^*\tilde{D}(a)\chi, \quad \text{hence} \quad D'(a) = S^{-1}D(a)S^* \tag{16}$$

Two corepresentations of the group $\mathcal{G}$, the corepresentation with the matrices $D(g)$ and $D(a)$, and the corepresentation with the matrices $D'(g)$ and $D'(a)$, are said to be equivalent if there exists a nonsingular matrix S such that

$$D'(g) = S^{-1}D(g)S; \quad \forall g \in G, \tag{17}$$

$$D'(a) = S^{-1}D(a)S^*; \quad \forall a \in a_0G \tag{18}$$

We envisage the following similarity transformations of the corepresentation matrices:

$$S_1 = e^{-i\alpha_0/2}E, \qquad S_2 = e^{-i\alpha_0/2}\left(\begin{array}{c|c} E_d & 0 \\ \hline 0 & -E_d \end{array}\right), \qquad S_3 = e^{-i\alpha_0/2}\left(\begin{array}{c|c} 0 & E_d \\ \hline \pm E_d & 0 \end{array}\right) \tag{19}$$

which depend on a real parameter α_0, where E is the unit matrix of an appropriate dimension and E_d is the unit matrix of dimension d. These transformations preserve the block structure of the corepresentation matrices. The matrices $D'(g)$ remain independent of the parameter α_0 under these transformations, while the matrices $D'(a)$ acquire the factor $\exp(i\alpha_0)$. These transformations will be considered for the definition of the matrix algebras in Sect. 3.

2.2 Reducibility of Corepresentations

If the basis χ in (14) can be transformed by a nonsingular transformation S so that the new basis $\chi' = \tilde{S}\chi$ is the direct sum of two subspaces which are both invariant under the group $\mathcal{G}$, the corep $D\Gamma$ is said to be *reducible*. If not, $D\Gamma$ is said to be *irreducible*. We observe that we use the term *reducible*, as it is used in [4, 29, 32], in the sense of *completely reducible* [5], or *decomposable* [2]. The two representations Γ and $\overline{\Gamma}$ may be inequivalent or equivalent. The answer to the question about the reducibility of the corep in (11) and (12) hinges upon that.

2.2.1 The Representations Γ and $\overline{\Gamma}$ are Inequivalent

If the irreps Γ and $\overline{\Gamma}$ are inequivalent, the corep of the group $\mathcal{G}$ derived from the irrep Γ is irreducible. We are dealing with c-type irreducible corepresentation (type 3 in [32]), with the matrices in (11) and (12). The respective proof for irreps Γ which need not be unitary is analogous to that for unitary irreps Γ (for the later see [4]).

2.2.2 The Representations Γ and $\overline{\Gamma}$ are Equivalent

There exists then a nonsingular matrix N (the matrix β in [32]) such that

$$\Delta(g) = N\Delta^*(a_0^{-1}ga_0)N^{-1}, \quad \forall g \in G \tag{20}$$

Replacing the element g with $a_0^{-1}ga_0$ we also obtain

$$\Delta^*(a_0^{-1}ga_0) = N^*\Delta(a_0^{-2}ga_0^2)(N^{-1})^* = N^*\Delta(a_0^{-2})\Delta(g)\Delta(a_0^2)(N^{-1})^* \tag{21}$$

Substituting the last expression into (20) we obtain the equation

$$\Delta(g) = NN^*\Delta^{-1}(a_0^2)\Delta(g)\Delta(a_0^2)(N^{-1})^*N^{-1}, \quad \forall g \in G \tag{22}$$

Since Γ is irreducible, it follows from Schur's Lemma that $NN^*\Delta^{-1}(a_0^2) = \Lambda E$ where Λ is a constant and E is the unit matrix. Hence we obtain:

$$\Delta(a_0^2) = \Lambda^{-1}NN^*, \quad \text{and} \quad \Delta^*(a_0^2) = (\Lambda^*)^{-1}N^*N \tag{23}$$

In (20) we can put $g = a_0^2$ and we then obtain

$$\Delta(a_0^2) = N\Delta^*(a_0^2)N^{-1} \tag{24}$$

Substituting the right hand sides of (23) into (24), we obtain the equalities: $\Lambda^{-1}NN^* = N(\Lambda^*)^{-1}N^*NN^{-1} = (\Lambda^*)^{-1}NN^*$, and hence $\Lambda = \Lambda^*$. Calculating the determinant of both sides of (23) we obtain:

$$\Lambda = \pm\frac{|\det N \det N^*|}{|\det\Delta(a_0^2)|} = \pm 1 \tag{25}$$

when we assume that the irrep Γ consists of matrices with $|\det\Delta(g) = 1|$, and we remember that the matrix N can always be chosen so as to have $|\det N \det N^*| = 1$. Consequently, from (23) we obtain

$$NN^* = \pm\Delta(a_0^2) \tag{26}$$

as in the case of unitary matrices N, as in [4, 13, 32]. The reducibility of a corep depends on the sign in (26).

A corepresentation $D\Gamma$ is reducible if and only if the matrices $D(g)$ and $D(a)$ can simultaneously be expressed in the same block-diagonal form. The matrices $D(g)$ in (11) are already in a reduced form, however, it will be convenient to convert them to the form, when there are the same blocks along the diagonal. Applying the matrix

$$W = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & -N^{-1} \end{array}\right) \tag{27}$$

with N from (20), and $D(a_0)$ in (12), we obtain:

$$D'(g) = W^{-1}D(g)W = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta(g) \end{array}\right) \tag{28}$$

and

$$D'(a_0) = W^{-1}D(a_0)W^* = \left(\begin{array}{c|c} 0 & -\Delta(a_0^2)(N^{-1})^* \\ \hline -N & 0 \end{array}\right) \tag{29}$$

Since every element of $\mathcal{G}$ is of the form g, a_0g or ga_0, for $g \in G$, while $D'(a_0g) = D'(a_0)D'^*(g)$ and $D'(ga_0) = D'(g)D'(a_0)$, a nonsingular transformation V is required, which will reduce the matrices $D'(a_0)$ to block-diagonal form, leaving the matrices $D'(g)$ unaltered. That V must commute with $D'(g)$ in (28). Writing:

$$V^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{30}$$

from the equation $V^{-1}D'(g) = D'(g)V^{-1}$ we obtain:

$$\begin{pmatrix} \alpha\Delta(g) & \beta\Delta(g) \\ \gamma\Delta(g) & \delta\Delta(g) \end{pmatrix} = \begin{pmatrix} \Delta(g)\alpha & \Delta(g)\beta \\ \Delta(g)\gamma & \Delta(g)\delta \end{pmatrix} \tag{31}$$

As the matrices $\Delta(g)$ are irreducible, from Schur's Lemma we find that $\alpha = \lambda E$, $\beta = \mu E$, $\gamma = \nu E$ and $\delta = \rho E$, with constant λ, μ, ν, ρ, where E is a d-dimensional unit matrix. We therefore must have

$$V^{-1} = \begin{pmatrix} \lambda E & \mu E \\ \nu E & \rho E \end{pmatrix} \tag{32}$$

The required existence of V implies that $\det V^{-1} \neq 0$, which leads to

$$\lambda\rho \neq \mu\nu \tag{33}$$

We find that

$$V = \frac{1}{2}\begin{pmatrix} E/\lambda & E/\nu \\ E/\mu & E/\rho \end{pmatrix} \tag{34}$$

with

$$\lambda\rho = -\mu\nu \tag{35}$$

which is the condition for a reduction of a corep to be possible. It is the same as for unitary irreps Γ of the group G. With $D'(a_0)$ in (29), the transformed matrix $D''(a_0)$ has the form

$$\begin{aligned} &D''(a_0) \\ &\quad = V^{-1}D'(a_0)V^* \\ &\quad = \frac{1}{2}\left(\begin{array}{c|c} -(\mu/\lambda^*)N - (\lambda/\mu^*)\Delta(a_0^2)(N^{-1})^* & -(\mu/\nu^*)N - (\lambda/\rho^*)\Delta(a_0^2)(N^{-1})^* \\ \hline -(\rho/\lambda^*)N - (\nu/\mu^*)\Delta(a_0^2)(N^{-1})^* & -(\rho/\nu^*)N - (\nu/\rho^*)\Delta(a_0^2)(N^{-1})^* \end{array}\right) \end{aligned} \tag{36}$$

As the off-diagonal terms have to vanish, and from (35), we obtain the condition:

$$NN^* = \frac{|\lambda|^2}{|\mu|^2}\Delta(a_0^2) \tag{37}$$

which has the form of (26) with $(+)$ sign, provided that

$$\frac{|\lambda|^2}{|\mu|^2} = 1 \tag{38}$$

2.2.3 *The Irreps* Γ *and* $\overline{\Gamma}$ *Are Equivalent and a Reduction of the Corepresentation in* (11) *and* (12) *Is Possible*

Considering (37) and (38) we find that a reduction of the corep in (11) and (12) is possible when

$$NN^* = +\Delta(a_0^2) \tag{39}$$

Taking into account (35) and (37), we obtain from (36) the reduced matrix $D''(a_0)$ in the form

$$D''(a_0) = \left(\begin{array}{c|c} (-\mu/\lambda^*)N & 0 \\ \hline 0 & (-\rho/\nu^*)N \end{array}\right) \tag{40}$$

Owing to (35), the coefficients μ/λ^* and ρ/ν^*, have the same absolute value and they can differ only by a phase factor, and hence, according to (18), the two blocks along the diagonal are equivalent. For a unitary N, the matrix in (40) turns into the customary matrix $D''(a_0)$, for example in Eq. (7.3.40) of [4], or in Eq. (1.5.40) of [13].

In order to determine the matrix connected with the element $a = ga_0$ we use the second from (13) and obtain $D(a) = D(ga_0) = D(g)D(a_0)$, hence from (28) and (40) we obtain the matrix

$$D''(ga_0) = \left(\begin{array}{c|c} -(\mu/\lambda^*)\Delta(g)N & 0 \\ \hline 0 & -(\rho/\nu^*)\Delta(g)N \end{array}\right) \tag{41}$$

We observe that the reduced matrices in (28) and (41), with the two blocks in $D''(a)$ in the same form, can be obtained by applying to corep matrices in (11) and (12) the transformation, which is analogous to that given by Kovalev and Gorbanyuk for unitary groups [29], namely:

$$V_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} E & iE \\ \hline (\lambda/\mu)N^{-1} & -i(\lambda/\mu)N^{-1} \end{array}\right), \qquad V_1^{-1} = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} E & (\mu/\lambda)N \\ \hline -iE & i(\mu/\lambda)N \end{array}\right) \tag{42}$$

Applying this transformation we obtain the corep matrices in the form:

$$D'(g) = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta(g) \end{array}\right), \quad D'(ga_0) = \left(\begin{array}{c|c} (\mu/\lambda)\Delta(g)N & 0 \\ \hline 0 & (\mu/\lambda)\Delta(g)N \end{array}\right) \tag{43}$$

and

$$D'(a_0 g) = \left(\begin{array}{c|c} (\mu/\lambda)N\Delta^*(g) & 0 \\ \hline 0 & (\mu/\lambda)N\Delta^*(g) \end{array}\right) \tag{44}$$

With $g = E$, we obtain

$$D'(a_0) = \left(\begin{array}{c|c} (\mu/\lambda)N & 0 \\ \hline 0 & (\mu/\lambda)N \end{array}\right) \tag{45}$$

which replaces (40), in which the two block matrices appear with opposite signs. When the matrix N is unitary, and we put $\mu/\lambda = 1$, the transformation V_1 in (42) turns into Eq. (8.11a) in [29], or into Eq. (1.5.43) in [13]. In general we have $\mu/\lambda = \exp(i\xi)$, with a real ξ. Renaming the functions ϕ_i in (4) of the original corepresentation,

$$\phi_i = a_0\varphi_i = \varphi_{d+i}, \quad i = 1, \ldots, d \tag{46}$$

and utilizing the transformation V_1 in (42), we obtain the basis functions of the two blocks, with labels (1) and (2),

$$\psi_j^{(1)} = \frac{1}{\sqrt{2}}\left(\varphi_j + \frac{\lambda}{\mu}\sum_{i=1}^{d}(\tilde{N}^{-1})_{ji}\varphi_{d+i}\right), \qquad \psi_j^{(2)} = \frac{i}{\sqrt{2}}\left(\varphi_j - \frac{\lambda}{\mu}\sum_{i=1}^{d}(\tilde{N}^{-1})_{ji}\varphi_{d+i}\right) \tag{47}$$

In the case of unitary groups G, when the original basis functions φ_j in (2) are orthogonal, the basis functions $\psi_j^{(1)}$, $j = 1, \ldots, d$, also are orthogonal, and the same holds for the functions $\psi_j^{(2)}$. These two sets of functions need not be mutually orthogonal, however.

2.2.4 *The Irreps* Γ *and* $\overline{\Gamma}$ *Are Equivalent, However a Reduction of the Corepresentation in* (11) *and* (12) *Is Impossible*

According to (26) with the $(-)$ sign, we now have:

$$NN^* = -\Delta(a_0^2) \tag{48}$$

and from (29) we obtain

$$D'(a_0) = \left(\begin{array}{c|c} 0 & N \\ \hline -N & 0 \end{array}\right) \tag{49}$$

With $a = ga_0$, hence $g = aa_0^{-1}$, and $D'(a) = D'(g)D'(a_0)$, with $D'(g)$ in (28) and $D'(a_0)$ in (49), we obtain

$$D'(ga_0) = \left(\begin{array}{c|c} 0 & \Delta(g)N \\ \hline -\Delta(g)N & 0 \end{array}\right) \tag{50}$$

With $a = a_0 g$, hence $g = a_0^{-1}a$, from the third of (13) and from (49) we obtain the matrix

$$D'(a_0 g) = D'(a_0)D'^*(g) = \left(\begin{array}{c|c} 0 & N\Delta^*(g) \\ \hline -N\Delta^*(g) & 0 \end{array}\right) \tag{51}$$

We observe that (28), (50) and (51) can be obtained by applying to the corep matrices in (11) and (12) the transformation:

$$V_2 = \left(\begin{array}{c|c} iE & 0 \\ \hline 0 & iN^{-1} \end{array}\right) \tag{52}$$

When the matrix N is unitary, V_2 turns into the transformation given by Kovalev and Gorbanyuk in Eq. (8.10) of [29], (or Eq. (1.5.50) of [13]).

The basis functions transforming according to the matrices in (28), (50) and (51) are determined from the equality,

$$\tilde{V}_2 \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} i\varphi \\ i\tilde{N}^{-1}\phi \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2d} \end{pmatrix} \tag{53}$$

with ϕ_i in (46) or,

$$\begin{aligned} &\psi_j = i\varphi_j, \quad j = 1, \ldots, d \\ &\psi_{d+j} = i\sum_{k=1}^{d} (\tilde{N}^{-1})_{jk}\varphi_{d+k}, \quad j = 1, \ldots, d, \quad \text{with} \quad \varphi_{d+k} = a_0\varphi_k \end{aligned} \tag{54}$$

The corepresentation formulas hold for single-valued as well as for double-valued representations Γ of the subgroup G.

3 Continuous Groups with Antilinear Operations

Wigner [32] considered continuous groups with antilinear operations $G + a_0G$, where the group G is unitary and $a_0^2 \in G$. In the following definition, the group G will be a linear Lie group which need not be unitary, and $a_0^2 = \pm 1$.

Definition 3.1 A type of continuous groups with antilinear operations has the form $G + a_0G$, where G is a linear Lie group and a_0 is an antilinear operation which fulfils the condition $a_0^2 = \pm\mathbf{1}$.

Before formulating the conditions which any group $G + a_0G$ defined above has to fulfill, we firstly will discuss the problem of the parametrization of the coirreps of that group, and next the problem of the matrix algebra with the commutator product, connected with the above defined group. In the following the name "Lie group" is used in the sense of "linear Lie group".

Observation 3.1 *From* (11) *and* (12) *it is seen that the matrices $D(g)$ and $D(a)$ of the corepresentation $D\Gamma$, depend on n essential parameters $\alpha_1, \ldots, \alpha_n$ of the subgroup G. However, for completing the basis of the algebra connected with the group $G + a_0G$, with the commutator product, the basis element connected with the antilinear element a_0 can be required. That basis element can be determined when the corepresentation matrices of the coset a_0G depend on an additional parameter. Such a parameter can be introduced owing to the form of the equivalence condition of two corepresentations in* (17) *and* (18). *The matrices $D'(a)$ then depend on the n essential parameters of the group G and on the parameter α_0. Consequently, the coirrep $D\Gamma$ depends on n independent parameters $\alpha_1, \ldots, \alpha_n$ of the linear Lie group G, and on the additional parameter α_0, which appears only in the matrices of the coset a_0G. We will apply the transformation S_1 in* (19).

It seems that the necessity of introducing the additional parameter is connected with the existence of two convergence points in the groups $G + a_0G$. In Lie groups, when the essential parameters approach zero values, the representation matrices converge to the unit matrix. In $G+a_0G$ groups there are two points of convergence: the unit matrix E for the Lie group G matrices, and the matrix $D(a_0)$ for the matrices of the coset a_0G. In other words, while the local properties of a Lie group G are connected with a small vicinity of the identity transformation, the local properties of the group $G+a_0G$ are connected with the vicinities of two transformations: the identity transformation and the transformation connected with the antilinear element a_0.

Depending on the particular group $G+a_0G$, the basis element X_0' connected with the matrix $e^{i\alpha_0} D(a_0)$ either commutes or it does not commute with the remaining basis elements of the respective algebra. The question can therefore be posed whether it is legitimate to retain the parameter α_0, with which X_0' is connected, when the algebra is complete without X_0'. Two answers of this question can be considered: (1) The presence of an additional parameter in the coset a_0G corepresentation matrices is implied by the necessity of completing the algebra. When it turns out that the algebra is complete without the basis element X_0', there is no need for an additional parameter. There is then no basis element of the algebra which is connected with the antilinear operation a_0. (2) Any corepresentation can be transformed to the form with the matrices of the coset a_0G depending on the additional parameter. This parameter leads to the basis element X_0', of the matrix algebra which is connected with the antilinear operation a_0.

If we accepted the first answer, semi-simple algebras for the Lie group G could turn into semi-simple algebras for $G+a_0G$. The acceptance of the second answer implies that when X_0' commutes with all the remaining basis elements, semi-simple algebras of the Lie group G do not turn into semi-simple algebras of the group $G+a_0G$. In the following we will accept α_0 as an essential parameter of the coirrep connected with the group $G+a_0G$, also in the case when X_0' is not required for completing the basis of the matrix algebra. This means that we accept the second standpoint.

Considering Observation 3.1 we introduce the following four conditions which have to be fulfilled by any continuous group with antilinear operations, in the sense of Definition 3.1.

(I) The group $G+a_0G$ must have at least one faithful finite-dimensional irreducible corepresentation $D\Gamma$ of type (a) or (b), of dimension $(n+1)$, i.e. with $(n+1)$ essential parameters $\alpha_0, \alpha_1, \dots, \alpha_n$.

Let the dimension of the irrep matrices Δ of the subgroup G be d. The dimension of the coirrep $D\Gamma$ is d or $2d$, for a-type or b-type coirreps, respectively.

We define the distance function denoted by $d_1(g,g')$ between the elements g and g' of G, and the distance function $d_2(a,a')$ between the elements a and a' of the coset a_0G,

$$d_1(g,g') = +\left[\sum_{j=1}^{m}\sum_{k=1}^{m}|D(g)_{jk}-D(g')_{jk}|^2\right]^{1/2}$$

$$d_2(a,a') = +\left[\sum_{j=1}^{m}\sum_{k=1}^{m}|D(a)_{jk}-D(a')_{jk}|^2\right]^{1/2} \tag{55}$$

where $m=d$, for a-type coirreps, and $m=2d$, for b-type coirreps. The parameter α_0 does not appear in the distance function $d_2(a,a')$. These distance functions fulfill the following five conditions:

(1) $d_1(g,g')=d_1(g',g), \quad d_2(a,a')=d_2(a',a)$

(2) $d_1(g,g)=0, \quad d_2(a,a)=0$

(3) $d_1(g, g') > 0$, if $g \neq g'$ and $d_2(a, a)' > 0$ if $a \neq a'$

(4) $d_1(g, g'') \leq d_1(g, g') + d_1(g', g'')$ and $d_2(a, a'') \leq d_2(a, a') + d_2(a', a'')$

for any three elements of G or of a_0G. (56)

The two sets of elements g of G and a of a_0G which fulfill the conditions

$$d_1(g, \mathbf{1}) < \delta_1, \qquad \text{and} \qquad d_2(a, a_0) < \delta_2 \tag{57}$$

respectively, where δ_1 and δ_2 are real positive numbers, are said to be within the sphere of radius δ_1 centered on the unit element $\mathbf{1}$, and to be within the sphere of radius δ_2 centered on the element a_0, respectively. We are dealing with two small neighbourhoods of $\mathbf{1}$ and of a_0, respectively. The parameters $\alpha_1, \ldots, \alpha_n$, on which depend the matrices $D(g)$, representing the Lie subgroup G, are assigned to an n-dimensional Euclidean space $\mathcal{R}^n$, and the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$, on which depend the matrices $e^{i\alpha_0}D(a)$, representing the coset a_0G, are assigned to an $(n+1)$-dimensional Euclidean space $\mathcal{R}^{(n+1)}$.

(II) We fix a $\delta_1 > 0$ in $\mathcal{R}^n$, and we consider elements g of G lying in the sphere of radius δ_1 centered on the unit element $\mathbf{1}$. At the same time we fix a $\delta_2 > 0$ in $\mathcal{R}^{(n+1)}$, and consider elements a of a_0G, lying in the sphere of radius δ_2 which is centered on the element a_0. The elements $g \in G$ within the sphere of radius δ_1 are uniquely parametrized by n real parameters $\alpha_1, \ldots, \alpha_n$, and the elements $a \in a_0G$ are uniquely parametrized by the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$, when α_0 and $\alpha_0 + 2\pi p$, $p = \pm 1, 2, \ldots,$ are identified. The matrix E, representing the unit element $\mathbf{1}$, and the matrix $e^{i\alpha_0}D(a_0)$, representing the antilinear element a_0 are connected with $\alpha_1 = \cdots = \alpha_n = 0$.

(III) There has to exist such $\epsilon_1 > 0$, that to every point in $\mathcal{R}^n$ for which

$$\sum_{j=1}^{n} \alpha_j^2 < \epsilon_1^2 \tag{58}$$

there corresponds some element g, and there has to exist such $\epsilon_2 > 0$, that to every point in $\mathcal{R}^{(n+1)}$ for which

$$\sum_{j=1}^{n} \alpha_j^2 < \epsilon_2^2, \qquad \text{with a fixed } \alpha_0 \tag{59}$$

there corresponds some element $a = a_0 g$. There is a one-to-one correspondence between elements g of G, and points in $\mathcal{R}^n$, as well as between elements a of a_0G and points in $\mathcal{R}^{(n+1)}$, (provided that α_0 is identified with $\alpha_0 + 2\pi p$, $p = 1, 2, \ldots$), satisfying the respective condition in (58) or (59).

(IV) Each of the matrix elements of the coirrep $D\Gamma(\alpha_0, \alpha_1, \ldots, \alpha_n)$ must be an analytic function of the parameters $(\alpha_1, \ldots, \alpha_n)$ for the subgroup G, and of these parameters together with the parameter α_0 for the coset a_0G. These parameters have to satisfy the respective conditions in (58) and (59) This means that for the subgroup G, each of the matrix elements D_{jk} can be expressed as a power series in $\alpha_1 - \alpha_1^0, \ldots, \alpha_n - \alpha_n^0$, for all $(\alpha_1^0, \ldots, \alpha_n^0)$ fulfilling the condition in (58), and for the coset a_0G, each of the matrix elements $\exp(i\alpha_0)D_{jk}$ can be expressed as a power series in $\alpha_0 - \alpha_0^0, \alpha_1 - \alpha_1^0, \ldots,$ $\alpha_n - \alpha_n^0$, for all $\alpha_0^0, \alpha_1^0, \ldots, \alpha_n^0$ fulfilling the condition in (59).

Consequently, all the derivatives: $\partial D_{jk}/\partial\alpha_p$, $\partial^2 D_{jk}/\partial\alpha_p\partial\alpha_q, \dots,$ for the subgroup, and $\partial\exp(i\alpha_0)D_{jk}/\partial\alpha_p$, $\partial^2\exp(i\alpha_0)D_{jk}/\partial\alpha_p\partial\alpha_q, \dots,$ for the coset, have to exist at all points which fulfill (58), and (59), including the points $\alpha_1 = \cdots = \alpha_n = 0$, and $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$ for the subgroup and for the coset, respectively. For a-type coirreps, we have $p, q = 1, \dots, n$ and $j, k = 1, \dots, d$, for the subgroup, and $p, q = 0, 1, \dots, n$, for the coset, and for b-type coirreps we have $p, q = 1, \dots, n$; $j, k = 1, \dots, 2d$, for the subgroup, and $p, q = 0, 1, \dots, n$; $j, k = 1, \dots, 2d$, for the coset.

Observation 3.2 *The definitions of this Section are valid for the "unprimed" form of the corepresentation matrices in* (11) *and* (12), *with the factor* $\exp(i\alpha_0)$ *in front of the* $D(a_0g)$ *matrices, as well as for the "primed" form of the corepresentation matrices obtained with the help of the transformations in* (42) *and* (52). *The "prime" label of the corepresentation matrices D has therefore been omitted in points I, II and IV and it will be omitted further on in this section.*

Definition 3.2 The connected component of the group $G + a_0G$, is the maximal set of elements g or a which can be obtained from each other by continuously varying one or more of the respective matrix elements $D(g)_{jk}$ or $\mathrm{e}^{i\alpha_0}D(a)_{jk}$ of the faithful finite-dimensional coirrep $D\Gamma$.

For both types of coirreps, a and b, there holds the equivalence condition in (20), from which we obtain the equality:

$$N\Delta^*(g') = \Delta(g)N \tag{60}$$

where $g' \in G$, and for $a_0 = K$ we have $g' = g^*$, and for $a_0 = \Theta$, $g' = \Theta^{-1}g\Theta$.

For a-type coireps, when $a_0 = K$, from (60) we obtain: $N\Delta(g) = \Delta(g)N$, and hence from Schur's lemma we can put $N = E$, with E denoting the unit matrix. The matrices $D'(g)$, $D'(Kg)$ and $D'(gK)$ reduce to single blocks, which are $\Delta(g)$, $\exp(i\alpha_0)\Delta^*(g)$, and $\exp(i\alpha_0)\Delta(g)$, respectively. These can be transformed into one another by a continuous variation of one or more of the essential parameters. If the Lie group G is connected, the group $G + a_0G$ also is connected. When $a_0 \neq K$ and $N \neq E$, we obtain two different matrices representing group elements: $\Delta(g)$ and $\exp(i\alpha)\Delta(g)N$. We cannot obtain $\Delta(g)$ from $\Delta(g)N$, by a continuous variation of one or more of the essential parameters. When the essential parameters approach zero value, the above two matrices converge to E and to $N \neq E$, respectively. The group $G + a_0G$ then is not connected.

For b-type coirreps, the matrices of the coset a_0G due to their form cannot be transformed into the matrices of the subgroup G by a continuous variation of one or more of the essential parameters. When the essential parameters $\alpha_0, \alpha_1, \dots, \alpha_n$ approach zero values, the matrices $D(g)$ converge to the unit matrix E, and the matrices $\mathrm{e}^{i\alpha_0}D(a_0g)$ and $\mathrm{e}^{i\alpha_0}D(ga_0)$ to the matrix $D(a_0)$ in (49). According to Definition 3.2, for b-type coirreps the groups $G + a_0G$ are not connected.

Definition 3.3 For a-type coirreps we define the d-dimensional matrices $X_1, \dots, X_n$, connected with the subgroup G, and the d-dimensional matrices $X'_{n+1}, \dots, X'_{2n}$ and X'_0, connected with the coset a_0G, by their elements

$$(X_p)_{jk} = \left(\frac{\partial D(g)_{jk}}{\partial\alpha_p}\right)_{\alpha_1=\cdots=\alpha_n=0}; \quad p = 1, \dots, n \tag{61}$$

$$(X'_q)_{jk} = \left(\frac{\partial e^{i\alpha_0} D(a_0 g)_{jk}}{\partial \alpha_q} \right)_{\alpha_0=\alpha_1=\cdots=\alpha_n=0}; \quad q = 0, 1, \ldots, n \tag{62}$$

where $D(g)_{jk}$ and $e^{i\alpha_0} D(a_0 g)_{jk}$, $j, k = 1, \ldots, d$ denote the elements of the respective coirrep matrices.

Corollary 3.1 *For a-type coirreps, when all the matrices X'_q, $q = 1, \ldots, n$, are linearly dependent on the matrices X_p, $p = 1, \ldots, n$,, the $(n+1)$ matrices $X'_0, X_1, \ldots, X_n$ span an $(n+1)$-dimensional vector basis.*

Proof The proof is analogous to that for Lie groups in [5]. □

Definition 3.4 For b-type coirreps, the $2d$-dimensional matrices $X_1, \ldots, X_n, X'_{n+1}, \ldots, X'_{2n}$, and X'_0 are defined by their elements

$$(X_p)_{jk} = \left(\frac{\partial D(g)_{jk}}{\partial \alpha_p} \right)_{\alpha_1=\alpha_2=\cdots=\alpha_n=0}; \quad p = 1, \ldots, n \tag{63}$$

$$(X'_q)_{jk} = \left(\frac{\partial e^{i\alpha_0} D(a_0 g)_{jk}}{\partial \alpha_q} \right)_{\alpha_0=\alpha_1=\cdots=\alpha_n=0}; \quad q = 0, 1, \ldots, n \tag{64}$$

where $e^{i\alpha_0} D(a_0 g)_{jk}$, $j, k = 1, 2, \ldots, 2d$, denote the elements of the coirrep matrices of the coset $a_0 G$, and where the matrices $D(a_0 g)$ do not depend on the parameter α_0.

Corollary 3.2 *For b-type coirreps, the matrices $X_1, \ldots, X_n, X'_{n+1}, \ldots, X'_{2n}$, and X'_0 defined by* (63) *and* (64) *span a $(2n+1)$-dimensional real vector space.*

Proof Because of their form, the matrices $X'_{(n+1)}, \ldots, X'_{(2n)}, X'_0$, connected with the elements a in $a_0 G$, always are linearly independent of the matrices $X_1, \ldots, X_n$, connected with the subgroup G. We know that the matrices $X_1, \ldots, X_n$, are linearly independent [5]. It suffices to demonstrate the linear independence of the $(n+1)$ matrices $X'_0, X'_{n+1}, \ldots, X'_{2n}$, connected with the coset $a_0 G$. We have to show that the only solution of the equation

$$\left(\sum_{j=n+1}^{2n} \lambda_j X'_j \right) + \lambda_0 X'_0 = 0, \quad \text{with all } \lambda\text{'s real} \tag{65}$$

is $\lambda_{n+1} = \lambda_{n+2} = \cdots = \lambda_{2n} = \lambda_0 = 0$. The respective proof is analogous to that in [5, 7], for a Lie group G. □

Conjecture 3.1 *In a complex algebra, the matrices $X_1, \ldots, X_n, X'_0$, span an $(n+1)$-dimensional vector space.*

Definition 3.5 For the matrices $X_1, \ldots, X_n, X'_{n+1}, \ldots, X'_{2n}$, and X'_0, we define the commutator products $[A, B] = AB - BA$.

For the Lie group G we have:

$$[X_p, X_q] = \sum_{r=1}^{n} \overline{c}^{\,r}_{pq} X_r, \quad p, q, r = 1, \ldots, n \tag{66}$$

where $\overline{c}^{\,r}_{pq}$ are the structural constants. For the remaining commutator products we introduce.

Definition 3.6 The commutator of two basis vectors connected with the coset a_0G is equal to a linear combination of basis vectors connected with the subgroup G in [5, 10]

$$[X'_p,\ X'_q] = \sum_{r=1}^{n} \overline{d}^{\,r}_{pq} X_r, \quad p, q = 0, n+1, \ldots, 2n;\ r = 1, \ldots, n \tag{67}$$

and the commutator of a basis vector X_p, connected with the subgroup G, with a basis vector X'_q, connected with the coset a_0G, is equal to a linear combination of basis vectors connected with that coset,

$$[X_p, X'_q] = \sum_r \overline{e}^{\,r}_{pq} X'_r, \quad p = 1, \ldots, n;\ q = 0, n+1, \ldots, 2n;\ r = 0, n+1, \ldots, 2n \tag{68}$$

where the structural constants $\overline{d}^{\,r}_{pq}$ and $\overline{e}^{\,r}_{pq}$ are antisymmetric with respect to the interchange of the indices p and q.

The definitions in (66), (67) and (68) establish a correspondence between the results of products of elements in the group $G + a_0G$, and the results of the respective commutator products of the basis elements of the matrix algebra.

Corollary 3.3 *From the Jacobi identity for the double commutator* $[[X_p, X_q], X_r]$, *in Lie algebras we obtain the known relation between the structural constants* $\overline{c}^{\,r}_{pq}$ *in* (66). *From the three Jacobi identities connected with the double commutators*: $[[X_p, X_q], X'_r]$, $[[X_p, X'_q], X'_r]$ *and* $[[X'_p, X'_q], X'_r]$, *we obtain on the basis of* (66), (67) *and* (68) *the respective three relations between the structural constants* $\overline{c}^{\,s}_{pq}$, $\overline{d}^{\,s}_{pq}$ *and* $\overline{e}^{\,s}_{pq}$:

$$\begin{aligned}
\overline{c}^{\,s}_{pq}\overline{e}^{\,t}_{sr} - \overline{e}^{\,s}_{qr}\overline{e}^{\,t}_{ps} + \overline{e}^{\,s}_{pr}\overline{e}^{\,t}_{qs} &= 0 \\
\overline{e}^{\,s}_{pq}\overline{d}^{\,t}_{sr} + \overline{d}^{\,s}_{qr}\overline{c}^{\,t}_{sp} - \overline{e}^{\,s}_{pr}\overline{d}^{\,t}_{sq} &= 0 \\
\overline{d}^{\,s}_{pq}\overline{e}^{\,t}_{sr} + \overline{d}^{\,s}_{qr}\overline{e}^{\,t}_{sp} + \overline{d}^{\,s}_{rp}\overline{e}^{\,t}_{sq} &= 0
\end{aligned} \tag{69}$$

4 The Groups $SU(d) + KSU(d)$

We will apply the above presented theory to the unitary groups $SU(d)$, $d = 1, 2, \ldots$, considering the groups $SU(d) + KSU(d)$, where K denotes the operation of complex conjugation. We denote the matrices of the group $SU(d)$ by $\Delta(g)$. The group element $g(\alpha_1, \ldots, \alpha_n)$, of which $\Delta(g)$ is the matrix representation is written in the form,

$$g(\alpha_1, \ldots, \alpha_n) = \exp\left(i \sum_{j=1}^{n} \lambda_j \alpha_j\right) \tag{70}$$

where λ_j are real or imaginary symbols which fulfil the same commutation relations as the matrices representing them in the representation $SU(d)$, and α_j are real parameters. We have

$$Kg(\alpha_1, \ldots, \alpha_n) = g^* = \exp\left(-i \sum_{j=1}^{n} \lambda^*_j \alpha_j\right) \tag{71}$$

We consider the matrix $\overline{\Delta}(g)$ in (7), with $a_0 = K$, and we find that

$$\overline{\Delta}(g) = \Delta^*(K^{-1}gK) = \Delta^*(g^*K^2) = \Delta(g) \tag{72}$$

since $K^2 = 1$. The matrix N in (20), has the form

$$N = E \tag{73}$$

where E is a d-dimensional unit matrix. The reducibility condition: $NN^* = +\Delta(a_0^2)$ is fulfilled, since $\Delta(K^2) = E$ and $N = E$. Consequently, the coirreps of the groups $SU(d) + KSU(d)$ all are of a-type. The coirrep matrices are given in (43) and (44). The equivalence conditions of two coirreps in (17) and (18) allow for introducing the similarity transformation S_1 in (19) and hence the factor $\exp(i\alpha_0)$, with a real α_0 appears in front of the coset matrices $D(a)$. Consequently, we obtain the a-type coirrep matrices for the groups $SU(d) + KSU(d)$, in the form

$$D'(g) = \Delta(g), \quad D'(gK) = e^{i\alpha_0}\Delta(g) \quad D'(Kg) = e^{i\alpha_0}\Delta^*(g), \quad D'(K) = e^{i\alpha_0}E \tag{74}$$

The matrices $D'(g)$ depend on n parameters $(\alpha_1, \ldots, \alpha_n)$, and the matrices $D'(Kg)$ and $D'(gK)$, depend on $(n+1)$ parameters $(\alpha_0, \alpha_1, \ldots, \alpha_n)$.

We consider the real algebra of matrices X_σ connected with the subgroup G, and of matrices X'_ρ, connected with the coset a_0G,

$$X_\sigma = \left(\frac{\partial D(g)}{\partial \alpha_\sigma}\right)_{\alpha=0}, \quad X'_\rho = \left(\frac{\partial e^{i\alpha_0} D(a)}{\partial \alpha_\rho}\right)_{\alpha=0} \tag{75}$$

where for X_σ we have $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$, and for X'_ρ we have $\alpha \equiv (\alpha_0, \alpha_1, \ldots, \alpha_n)$. The matrices X'_ρ, connected with the coset a_0G, are linearly dependent on the matrices X_σ, except for the matrix X'_K, connected with α_0. In a real algebra we are dealing with $n+1$ basis elements. The algebra spanned by the matrices X_σ and X'_ρ differs from the algebra spanned by the symbols λ_j only in the presence in it of the matrix $X'_0 = iE$, which commutes with all the remaining matrices. In particular, the real algebra of $SU(2) + KSU(2)$ is the same as the algebra $su_L(2) \oplus u(1)$ which appears in the unified theory of weak and electromagnetic interactions [5].

The basis functions, which transform according to the coirrep in (43) and (44), are determined from (46) and (47), with $N = E$ and $\lambda/\mu = 1$, for unitary groups.

5 The Group $SL(2, C)$ Expressed in Terms of Cayley-Klein Parameters

We express the proper orthochronous Lorentz group $L_+^\uparrow$ in terms of the Cayley-Klein parameters as it was done in [30]. A four-vector $\mathbf{x}$ in the Minkowski space is written in the form:

$$\mathbf{x} = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_2 x_3 + \gamma_4 x_4 = \gamma_\mu x_\mu \tag{76}$$

with $x_4 = ict$, and under the group $L_+^\uparrow$ it is transformed according to the formula:

$$\mathbf{x}' = \gamma_\mu x'_\mu = S^{-1}(\gamma_\mu x_\mu)S \tag{77}$$

where S and S^{-1} are biquaternion transformations. Defining:

$$\gamma_{\mu\nu} := \gamma_\mu \gamma_\nu = -\gamma_{\nu\mu} \tag{78}$$

we have

$$S = A\gamma_{23} + B\gamma_{31} + C\gamma_{12} + D + ia\gamma_{14} + ib\gamma_{24} + ic\gamma_{34} - id\gamma_5 \tag{79}$$

where A, B, C, D, a, b, c, d are Cayley-Klein real parameters, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The inverse transformation S^{-1} is obtained from S by reversing the signs of the parameters A, B, C, a, b and c, which are called rotation parameters. We obtain $SS^{-1} = S^{-1}S = 1$ on two conditions:

$$A^2 + B^2 + C^2 + D^2 - a^2 - b^2 - c^2 - d^2 = 1, \tag{80}$$

$$Aa + Bb + Cc + Dd = 0 \tag{81}$$

which leave six independent Cayley-Klein rotation parameters, corresponding to the six rotation planes in the Minkowski space. The elements a_{jk} of the Lorentz matrix Λ in the expression:

$$x'_j = a_{jk} x_k \tag{82}$$

are calculated from (77) and (79) in the form [18, 19]:

$$\begin{aligned}
&a_{11} = (D^2 + A^2 - B^2 - C^2) + (d^2 + a^2 - b^2 - c^2)\\
&a_{22} = (D^2 - A^2 + B^2 - C^2) + (d^2 - a^2 + b^2 - c^2)\\
&a_{33} = (D^2 - A^2 - B^2 + C^2) + (d^2 - a^2 - b^2 + c^2)\\
&a_{44} = (D^2 + A^2 + B^2 + C^2) + (d^2 + a^2 + b^2 + c^2)\\
&a_{12} = 2[(AB + CD) + (ab + cd)], \quad a_{21} = 2[(AB - CD) + (ab - cd)]\\
&a_{13} = 2[(AC - BD) + (ac - bd)], \quad a_{31} = 2[(AC + BD) + (ac + bd)]\\
&a_{23} = 2[(BC + AD) + (bc + ad)], \quad a_{32} = 2[(BC - AD) + (bc - ad)]\\
&a_{14} = 2i[(Da - Bc) - (Ad - Cb)], \quad a_{41} = 2i[(Ad + Cb) - (Da + Bc)]\\
&a_{24} = 2i[(Ac - Bd) - (Ca - Db)], \quad a_{42} = 2i[(Ac + Bd) - (Ca + Db)]\\
&a_{34} = 2i[(Ba - Cd) - (Ab - Dc)], \quad a_{43} = 2i[(Ba + Cd) - (Ab + Dc)]
\end{aligned} \tag{83}$$

We next consider the irrep

$$\begin{aligned}
\gamma_1 &= \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & -i & \cdot \\ \hline \cdot & i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{array}\right), \quad
\gamma_2 = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \hline \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{array}\right), \\
\gamma_3 &= \left(\begin{array}{cc|cc} \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & i \\ \hline i & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot \end{array}\right), \quad
\gamma_4 = \left(\begin{array}{cc|cc} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{array}\right)
\end{aligned} \tag{84}$$

which is obtained when $\gamma_k = -i\beta\alpha_k$, $k = 1, 2, 3$, and $\gamma_4 = \beta$, where α_k and β are the matrices on page 368 of [8] or on page 121 of [9]. Applying to these matrices the unitary transformation [17],

$$Y = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{85}$$

we obtain the following irrep matrices

$$\gamma_1 = \left(\begin{array}{cc|cc} \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & -i \\ \hline -i & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{array}\right), \quad \gamma_2 = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \hline \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{array}\right),$$
$$\gamma_3 = \left(\begin{array}{cc|cc} \cdot & \cdot & \cdot & i \\ \cdot & \cdot & i & \cdot \\ \hline \cdot & -i & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot \end{array}\right), \quad \gamma_4 = \left(\begin{array}{cc|cc} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \hline 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{array}\right) \tag{86}$$

With this irrep of the γs, the transformation S in (79) takes the form:

$$S = \left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline 0 & \Delta_2 \end{array}\right) = \left(\begin{array}{cc|cc} \delta + i\alpha & -\beta + i\gamma & 0 & 0 \\ \beta + i\gamma & \delta - i\alpha & 0 & 0 \\ \hline 0 & 0 & \delta^* + i\alpha^* & -\beta^* + i\gamma^* \\ 0 & 0 & \beta^* + i\gamma^* & \delta^* - i\alpha^* \end{array}\right) \tag{87}$$

where

$$\alpha = A + ia, \quad \beta = B + ib, \quad \gamma = C + ic, \quad \delta = D + id \tag{88}$$

and where $*$ denotes a conjugate complex quantity.

In order to show that the matrices Δ_1 and Δ_2 constitute double-valued representations of the group $L_+^\uparrow$ we write the matrix form X of the four-vector $\mathbf{x}$ in (76),

$$X = \left(\begin{array}{cc|cc} 0 & 0 & x_4 + ix_1 & -x_2 + ix_3 \\ 0 & 0 & x_2 + ix_3 & x_4 - ix_1 \\ \hline x_4 - ix_1 & x_2 - ix_3 & 0 & 0 \\ -x_2 - ix_3 & x_4 + ix_1 & 0 & 0 \end{array}\right) \tag{89}$$

where we have used the matrix irrep for the γs in (86). Denoting the two off-diagonal blocks by X_{12} and X_{21}, we can verify that the transformation

$$SXS^\dagger = \left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline 0 & \Delta_2 \end{array}\right)\left(\begin{array}{c|c} 0 & X_{12} \\ \hline X_{21} & 0 \end{array}\right)\left(\begin{array}{c|c} \Delta_1^\dagger & 0 \\ \hline 0 & \Delta_2^\dagger \end{array}\right) = \left(\begin{array}{c|c} 0 & X'_{12} \\ \hline X'_{21} & 0 \end{array}\right) \tag{90}$$

where $\Delta^\dagger$ denotes the Hermitian conjugate matrix of the matrix Δ, leads to the two-to-one homomorphism of the matrices Δ_1 or Δ_2 onto the matrices Λ of the group $L_+^\uparrow$. Each of the matrices Δ_1 or Δ_2 in (87) constitutes the group $SL(2, C)$. The transformation S in (79) therefore determines the double-valued irrep $SL(2, C)$ of the group $L_+^\uparrow$.

The action of the time-reversal operation on the Cayley-Klein parameters can be determined while considering the transformation to a new coordinate system x_i', $i = 1, \ldots, 4$ with the time-reversal matrix T:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{91}$$

and transforming to that reference system the Lorentz matrix Λ in (83),

$$\Lambda' = T\Lambda T = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & -a_{14} \\ a_{21} & a_{22} & a_{23} & -a_{24} \\ a_{31} & a_{32} & a_{33} & -a_{34} \\ \hline -a_{41} & -a_{42} & -a_{43} & a_{44} \end{array}\right) \tag{92}$$

Comparing this expression for Λ' with Λ in (83), we find that the action of the time-reversal operation, denoted by Θ, on the Cayley-Klein parameters is determined by:

$$\Theta(A, B, C, D, a, b, c, d) = (A, B, C, D, -a, -b, -c, -d)\Theta \tag{93}$$

6 The Coirreps of the Groups $SL(2, C) + a_0 SL(2, C)$

We now return to the proper orthochronous Lorentz group constituted by the matrices Λ determined in (83). The metric in the Minkowski space is the object of which $L_+^\uparrow$ is the invariance group. The continuous group L' with antilinear operations is defined by:

$$L' = L_+^\uparrow + a_0 L_+^\uparrow \tag{94}$$

where a_0 is equal to the time-reversal operation Θ or to the operation of complex conjugation K.

6.1 The Coirrep of the Group $SL(2, C) + KSL(2, C)$

We will consider the matrix Δ_1 in (87). The following conclusions will also hold for the matrix Δ_2. We firstly have to answer the question whether the matrix $\overline{\Delta}_1(g) = \Delta_1^*(K^{-1}gK)$ is equivalent or inequivalent to the matrix $\Delta_1(g)$. With $\Delta_1(K^2) = E$, we obtain: $\Delta_1(K^{-1}gK) = \Delta_1(g^*K^2) = \Delta_1^*(g)$, with $\Delta_1^*(g) \in SL(2, C)$, and hence

$$\overline{\Delta}_1(g) = \Delta_1^*(K^{-1}gK) = \Delta_1(g) \tag{95}$$

and the matrices $\overline{\Delta}_1$ and Δ_1 are the same. We are dealing with a-type coirrep. The matrix N in the transformation $\Delta(g) = N\overline{\Delta}(g)N^{-1}$, in (20) is equal to

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{96}$$

The respective a-type coirrep is determined by a single diagonal block of the matrices in (43) and (44). We will retain the notation $D'(g)$ and $D'(a)$ also for the matrices of the coirrep consisting of single blocks, to indicate that these matrices are connected with the basis in (47). With N in (96), it follows from (37) that $|\lambda/\mu|^2 = 1$, and hence $\lambda/\mu = \exp(i\xi)$, with a real ξ. The factor $\exp(i\xi)$ can be absorbed by the factor $\exp(i\alpha_0)$, connected with the matrices $D'(Kg)$. In the matrix $D'(a) = D'(Kg)$, we therefore can write that $\lambda/\mu = 1$, and with $\Delta \equiv \Delta_1$ in (87), from (43) and (44) we obtain the coirrep matrices:

$$D'(g) = \Delta_1(g), \quad D'(gK) = \exp(i\alpha_0)\Delta_1(g), \quad D'(Kg) = \exp(i\alpha_0)\Delta_1^*(g) \tag{97}$$

6.2 The Coirrep of the Group $SL(2,C) + \Theta SL(2,C)$

We will show that the double-valued irrep $SL(2,C)$ of the group $L_+^\uparrow$, leads to a double-valued type-b coirrep of the group $L' = L_+^\uparrow + \Theta L_+^\uparrow$. To this end we will consider the matrix Δ_1 in (87). For the matrix Δ_2 the same conclusions will be valid. We have to answer the question whether the matrix $\overline{\Delta}_1(g) = \Delta_1^*(a_0^{-1} g a_0)$ is equivalent or inequivalent to the matrix $\Delta_1(g)$. Let $g(A,B,C,D,a,b,c,d)$ be an element of the group $L_+^\uparrow$, and $\Delta_1(g)$ be the matrix of the double-valued irrep $SL(2,C)$ of $L_+^\uparrow$, connected with that group element. With $a_0 = \Theta$, and $\Theta^2 = 1$, we obtain

$$\Delta_1(a_0^{-1} g a_0) = \Delta_1(\Theta g \Theta) = \Delta_1(g'\Theta^2) = \Delta_1(g'), \quad g' = \Theta g \Theta^{-1} \in L_+^\uparrow \tag{98}$$

and hence

$$\overline{\Delta}_1(g) = \Delta_1^*(a_0^{-1} g a_0) = \Delta_1^*(g') \tag{99}$$

where the matrix $\Delta_1^*(g')$ is obtained from the matrix $\Delta_1(g)$, by firstly reversing the signs of the parameters a, b, c, d, and thus obtaining the matrix $\Delta_1(g')$, and secondly by taking the complex conjugate of that matrix. It can be verified that the equivalence condition $\Delta_1(g) = N\overline{\Delta}_1(a_0^{-1} g a_0)N^{-1}$, which turns to: $\Delta_1(g)N = N\Delta_1^*(g')$, is fulfilled by the matrix:

$$N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{100}$$

We obtain $NN^* = -E$, with E denoting the two-dimensional unit matrix. The same conclusion holds for the irrep Δ_2. We therefore obtain for both matrices Δ_1 and Δ_2, the equality

$$NN^* = -E = -\Delta(\Theta^2) \tag{101}$$

Consequently, according to (48), the double-valued coirrep connected with the double-valued irrep Δ_1 or Δ_2, is of type b.

In order to determine the form of the coirrep matrices we utilize (28) and (49) through (51), the transformation S_1 in (19), with N in (100), we obtain:

$$\exp(i\alpha_0)D'(\Theta) = \exp(i\alpha_0)\begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix}, \quad D'(g) = \begin{pmatrix} \Delta_1(g) & 0 \\ 0 & \Delta_1(g) \end{pmatrix} \tag{102}$$

with $\Delta = \Delta_1$, in (87). Hence with $a_0 = \Theta$, and $\exp(i\alpha_0)D'(\Theta)$ in (102), for $a = a_0 g = \Theta g$, we find that

$$e^{i\alpha_0} D'(\Theta g) = \exp(i\alpha_0)D'(\Theta)D'^*(g) = \exp(i\alpha_0)\left(\begin{array}{c|c} 0 & N\Delta_1^*(g) \\ \hline -N\Delta_1^*(g) & 0 \end{array}\right)$$

$$= \exp(i\alpha_0)\left(\begin{array}{cc|cc} 0 & 0 & -\beta^*+i\gamma^* & -\delta^*-i\alpha^* \\ 0 & 0 & \delta^*-i\alpha^* & -\beta^*-i\gamma^* \\ \hline \beta^*-i\gamma^* & \delta^*+i\alpha^* & 0 & 0 \\ -\delta^*+i\alpha^* & \beta^*+i\gamma^* & 0 & 0 \end{array}\right) \quad (103)$$

with the respective expression for $e^{i\alpha_0}D'(g\Theta)$. This seems to be the first known b-type coirrep of a continuous group with antilinear operations.

7 The Matrix Algebras of the Groups $SL(2, C) + a_0SL(2, C)$, with $a_0 = K, \Theta$

7.1 The Group $SL(2, C) + KSL(2, C)$

The matrix basis elements for the real algebra are calculated from (61) and (62) and (97):

$$X_A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -X'_{KA}, \quad X_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = X'_{KB}, \quad X_C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -X'_{KC}$$
$$X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = X'_{Ka}, \quad X_b = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -X'_{Kb}, \quad X_c = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = X'_{Kc}$$
$$X'_K = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad (104)$$

which shows that the matrices X'_j, $j = 1, \ldots, 6$, connected with the coset a_0G are linearly dependent on the matrices X_j, $j = 1, \ldots, 6$, connected with the subgroup G. Consequently, in the real algebra, the number of basis elements is equal to the number of essential parameters. It is seen from (104) that in the respective complex algebra there are only four basis elements: X_A, X_B, X_C, X'_K.

We observe that the matrices of this complex algebra, which is connected with the group $SU(2, C) + KSU(2, C)$, are analogous to the matrices of the algebra $su_L(2) \oplus u(1)$, which appears in the unified theory of weak and electromagnetic interactions [5].

The commutator table of the basis elements of the matrix real algebra of the group $SL(2, C) + KSL(2, C)$ is given in Table 1.

The basis element X'_K is the center of the algebra. The algebra is not semi-simple. The structural constants defined in (66), (67) and (68) are determined by Table 1. It can be verified that they fulfill (69) in Corollary 3.3.

Table 1 The commutator table of the basis elements of the real matrix algebra of the group $SL(2, C) + KSL(2, C)$, with X_A replaced by A, X'_a by a, X'_K by K, with analogous abbreviations for the remaining basis elements

	A	B	C	a	b	c	K
A	0	$-2C$	$2B$	0	$-2c$	$2b$	0
B	$2C$	0	$-2A$	$2c$	0	$-2a$	0
C	$-2B$	$2A$	0	$-2b$	$2a$	0	0
a	0	$-2c$	$2b$	0	$2C$	$-2B$	0
b	$2c$	0	$-2a$	$-2C$	0	$2A$	0
c	$-2b$	$2a$	0	$2B$	$-2A$	0	0
K	0	0	0	0	0	0	0

7.2 The Group $SL(2,C)+\Theta SL(2,C)$

The matrix basis elements of the real algebra are calculated from (63), (64), (102) and (103). We obtain:

$$X_A=\left(\begin{array}{cc|cc} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ \hline 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\end{array}\right),\quad X_B=\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{array}\right),\quad X_C=\left(\begin{array}{cc|cc} 0 & i & 0 & 0\\ i & 0 & 0 & 0\\ \hline 0 & 0 & 0 & i\\ 0 & 0 & i & 0\end{array}\right)$$

$$X_a=\left(\begin{array}{cc|cc} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{array}\right),\quad X_b=\left(\begin{array}{cc|cc} 0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{array}\right),\quad X_c=-\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{array}\right)$$

$$X'_{\Theta A}=\left(\begin{array}{cc|cc} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ \hline 0 & i & 0 & 0\\ i & 0 & 0 & 0\end{array}\right),\quad X'_{\Theta B}=\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right),\quad X'_{\Theta C}=\left(\begin{array}{cc|cc} 0 & 0 & i & 0\\ 0 & 0 & 0 & -i\\ \hline -i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{array}\right)$$

$$X'_{\Theta a}=\left(\begin{array}{cc|cc} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ \hline 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right),\quad X'_{\Theta b}=\left(\begin{array}{cc|cc} 0 & 0 & i & 0\\ 0 & 0 & 0 & i\\ \hline -i & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{array}\right),\quad X'_{\Theta c}=\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ \hline -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right)$$

$$X'_{\Theta}=\left(\begin{array}{cc|cc} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ \hline 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{array}\right) \tag{105}$$

When $a=ga_0=g\Theta$, the matrices $X'_{a\Theta}$, $X'_{b\Theta}$, $X'_{c\Theta}$ acquire the opposite sign with respect to the matrices $X'_{\Theta a}$, $X'_{\Theta b}$, $X'_{\Theta c}$, while the remaining matrices are the same as in (105). The commutators of the basis elements of the real algebra, given in (105), are presented in Table 2.

It can be verified that the structural constants which are determined by Table 2, fulfill (69) in Corollary 3.3. We are dealing with two off-diagonal mutually-commuting matrices $X'_{\Theta B}$ and $X'_{\Theta b}=-iX'_{\Theta B}$, which commute with all other matrices in (105). These two matrices form the center of the real algebra of the group $SL(2,C)+\Theta SL(2,C)$. Consequently, this algebra is not semi-simple. The matrices $X'_{\Theta B}$ and $X'_{\Theta b}$ in (105) can be transformed to a diagonal form with the help of an appropriate matrix P. They then acquire the form:

$$\overline{X}'_{\Theta B}=P^{-1}X'_{\Theta B}P=\left(\begin{array}{cc|cc} i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ \hline 0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\end{array}\right) \tag{106}$$

and $\overline{X}'_{\Theta b}=-i\overline{X}'_{\Theta B}$. It is seen from (105) that the basis of the complex algebra consists of seven matrices: $X_A, X_B, X_C, X'_{\Theta A}, X'_{\Theta B}, X'_{\Theta C}, X'_{\Theta}$. The dimension of the complex algebra therefore is equal to the number of the essential parameters of the group $SL(2,C)+\Theta SL(2,C)$.

Table 2 The commutator table of the basis elements in (105), which constitute the $(2n+1)$-dimensional basis of the real algebra connected with the group $SL(2,C)+\Theta SL(2,C)$. We have replaced X_A by A, $X'_{\Theta A}$ by ΘA, X'_Θ by Θ, with analogous symbols for the remaining basis elements

	A	B	C	a	b	c	ΘA	ΘB	Θ C	Θa	Θ b	Θc	Θ
A	0	$-2C$	$2B$	0	$-2c$	$2b$	$2i\Theta$	0	0	2Θ	0	0	$-2\Theta a$
B	$2C$	0	$-2A$	$2c$	0	$-2a$	$2\Theta C$	0	$-2\Theta A$	$2\Theta c$	0	$-2\Theta a$	0
C	$-2B$	$2A$	0	$-2b$	$2a$	0	0	0	$2i\Theta$	0	0	2Θ	$-2\Theta c$
a	0	$-2c$	$2b$	0	$2C$	$-2B$	-2Θ	0	0	$2i\Theta$	0	0	$-2\Theta A$
b	$2c$	0	$-2a$	$-2C$	0	$2A$	$-2\Theta c$	0	$2\Theta a$	$2\Theta C$	0	$-2\Theta A$	0
c	$-2b$	$2a$	0	$2B$	$-2A$	0	0	0	-2Θ	0	0	$2i\Theta$	$-2\Theta C$
ΘA	$-2i\Theta$	$-2\Theta C$	0	2Θ	$2\Theta c$	0	0	0	$-2B$	0	0	$2b$	$2a$
ΘB	0	0	0	0	0	0	0	0	0	0	0	0	0
ΘC	0	$2\Theta A$	$-2i\Theta$	0	$-2\Theta a$	2Θ	$2B$	0	0	$-2b$	0	0	$2c$
Θa	-2Θ	$-2\Theta c$	0	$-2i\Theta$	$-2\Theta C$	0	0	0	$2b$	0	0	$2B$	$2A$
Θb	0	0	0	0	0	0	0	0	0	0	0	0	0
Θc	0	$2\Theta a$	-2Θ	0	$2\Theta A$	$-2i\Theta$	$-2b$	0	0	$-2B$	0	0	$2C$
Θ	$2\Theta a$	0	$2\Theta c$	$2\Theta A$	0	$2\Theta C$	$-2a$	0	$-2c$	$-2A$	0	$-2C$	0

8 Conclusions

Wigner [32] considered the continuous groups $G+a_0G$ with a unitary group G and the antilinear element a_0 with $a_0^2\in G$. The matrix algebras with commutator product connected with those groups were not considered. We have considered the continuous groups $G+a_0G$, where G is a linear Lie group, which need not be unitary, and a_0 is an antilinear operation which fulfills the condition $a_0^2=\pm 1$. The a-type and b-type irreducible corepresentations of the groups $G+a_0G$ were employed for the determination of the respective matrix algebras with commutator product. Some of the general properties of those algebras were determined.

We have presented the corepresentation theory without the assumption of the unitarity of the subgroup G of the group $G+a_0G$, where a_0 denotes an antilinear operation. This was done for coirreps of a-type or b-type (types 1 or 2, respectively, in [32]). To the matrices representing the elements of the coset a_0G, an additional parameter α_0 is assigned by means of the equivalence transformation of two coirreps. It then is possible to define the basis element X'_0, connected with the matrix $e^{i\alpha_0}D(a_0)$. This basis element in general is required for completing the algebra connected with the group $G+a_0G$. There are cases, however, depending on the Lie group G, and on the type of the antilinear element a_0, when X'_0 commutes with the remaining basis elements of the respective algebra. It then is not indispensable for completing that algebra. The parameter α_0 is included into the set of essential parameters of the group $G+a_0G$ in any case. Consequently, all the matrices $e^{i\alpha_0}D(a_0g)$, with $g\in G$, belong to an $(n+1)$-dimensional parameter space, while the matrices $D(g)$ belong to an n-dimensional parameter space of the Lie group G.

There appears a characteristic difference between the properties of Lie groups G, and of groups $G+a_0G$. In Lie groups, when the essential parameters approach zero values, the representing matrices converge to the unit matrix E. In $G+a_0G$ groups, there are two points of convergence: the unit matrix E for the Lie group G matrices, and the matrix $D(a_0)$ for the matrices of the coset a_0G. In other words, while the local properties of a Lie group G are connected with a small vicinity of the identity transformation, the local properties of the group $G + a_0G$ are connected with the vicinities of two transformations: the identity transformation and the transformation connected with the antilinear element a_0.

An application of this theory to the unitary groups $SU(d)$, $d = 1, 2, \dots$, showed that the coirreps of the groups $SU(d) + KSU(d)$ all are of a-type. The respective matrix algebras differ from the matrix algebras of the groups $SU(d)$ in the presence of the matrix $X'_0 = iE$, where E is the d-dimensional unit matrix. The b-type coirrep of the group $SL(2, C) + \Theta SL(2, C)$, where Θ is the time-reversal operation, seems to be the first known example of b-type coirrep of a continuous group with antilinear operations.

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